

Arbitrary p -Gradient Values

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Abstract

For any prime number p and any positive real number α , we construct a finitely generated group Γ with p -gradient equal to α . This construction is used to show that there exist uncountably many pairwise non-commensurable groups that are finitely generated, infinite, torsion, non-amenable, and residually- p .

1 Introduction

Let G be a finitely generated group and $d(G)$ denote the minimum number of generators of G . We start by recalling the Schreier Index Formula: Let H be a finite index subgroup of a finitely generated group G . Then $d(H) - 1 \leq (d(G) - 1)[G : H]$ and in particular if G is free of finite rank, then H is free and $d(H) - 1 = (d(G) - 1)[G : H]$. The rank gradient of a finitely generated group, in a sense, is a measure of how far the Schreier Index Formula is from being an equality rather than an inequality. From a group-theoretic point of view, this is already an interesting question. However, this was not the motivation that Mark Lackenby used to define rank gradient. Lackenby first introduced rank gradient as a means to study 3-manifold groups [8].

We define the absolute rank gradient of G by

$$RG(G) = \inf_{[G:H] < \infty} \frac{d(H) - 1}{[G : H]}$$

where the infimum is taken over all finite index subgroups H of G .

As we will see later, the rank gradient of a group is sometimes quite hard to calculate and work with in general. It is often more convenient to compute the rank gradient of the pro- p completion $G_{\hat{p}}$ of the group G for some fixed prime p . When dealing with profinite groups we use the notion

of topologically finitely generated instead of (abstractly) finitely generated. We can define the p -gradient of the group G , denoted $RG_p(G)$, as the rank gradient of $G_{\widehat{p}}$. We will also give a more explicit definition of p -gradient in Section 2.

Since Lackenby first defined rank gradient of a finitely generated group [8], the following conjecture has remained open.

Conjecture. *For every real number $\alpha > 0$ there exists a finitely generated group Γ such that $RG(\Gamma) = \alpha$.*

However, we are able to prove the analogous question for p -gradient.

Main Result. *For every real number $\alpha > 0$ and any prime p , there exists a finitely generated group Γ such that $RG_p(\Gamma) = \alpha$.*

Given a prime p and an $\alpha > 0 \in \mathbb{R}$, we start by considering a free group F of finite rank greater than $\alpha + 1$. We then take the set of all residually- p groups that are homomorphic images of F which have p -gradient greater than or equal to α . We partially order this set by $G_1 \succcurlyeq G_2$ if G_1 surjects onto G_2 . Then by a Zorn's Lemma argument, we show that this set has a minimal element, Γ . We show $RG_p(\Gamma) = \alpha$ by contradiction by constructing an element which is smaller than Γ . To construct this new smaller element, we use Theorem 5.2, which was proved using slightly different language and a different method by Barnea and Schlage-Puchta in [4], but is formulated and proved independently here as well.

The methods used to prove this result require and are similar to those used by Schlage-Puchta in his work on p -deficiency and p -gradient [17] and Osin in his work on rank gradient [14]. To prove the above set has a minimal element, we will use direct limits of groups and show how the p -gradient of each group in the direct limit relates to that of the p -gradient of the limit group. This idea (Lemma 5.5) was inspired by Pichot's similar result for L^2 -Betti numbers [15]. Along the way, we will also prove that the p -gradient of a group is equal to the p -gradient of its pro- p completion.

One of the main goals of Osin's [14] and Schlage-Puchta's [17] papers were to give a simple construction of non-amenable torsion residually finite groups. The construction given in this paper shows that there exist such groups with arbitrary p -gradient (Theorem 5.7). A simple consequence of this result is that there exist uncountably many pairwise non-commensurable groups that are finitely generated, infinite, torsion, non-amenable, and residually- p . The fact that the groups are non-commensurable uses the p -gradient and is almost immediate from the construction, which shows another way in which the p -gradient can be a useful tool.

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2 Rank Gradient and Other Group Invariants

Rank gradient is closely related to two other group invariants: L^2 -Betti numbers and cost. Lück proved that one can compute the first L^2 -Betti number of a finitely presented residually finite group G as follows [11].

Theorem 2.1. *Let G be a finitely presented residually finite group. Let $\{N_i\}$ be a descending chain of finite index normal subgroups of G such that $\bigcap_{i=1}^{\infty} N_i = \{1\}$. Then the first L^2 -Betti number of G is*

$$\beta_1^{(2)}(G) = \lim_{i \rightarrow \infty} \frac{rk(N_i^{ab})}{[G : N_i]},$$

where $rk(N_i^{ab})$ is the torsion free rank of the abelianization of N_i .

This value, $rk(N_i^{ab})$, is called the ordinary first Betti number of N_i and is typically denoted $b_1(N_i)$. It is important to note that Lück's calculation of $\beta_1^{(2)}(G)$ given above does not depend on the chain. The reader is referred to [12] for more information on Betti numbers.

The notion of cost was first introduced by Levitt [10]. For a definition of cost, the reader is referred to the article by Gaboriau in the November 2010 issue of Notices, titled "What is ... Cost?" [7].

If G is a finitely generated residually finite group, it is known that

$$RG(G) \geq \text{cost}(G) - 1 \geq \beta_1^{(2)}(G) - \frac{1}{|G|}.$$

Abert and Nikolov [2] proved the first part of the inequality and the second part was proved by Gaboriau [6].

It is not known whether or not the inequalities can be strict. The relationship between rank gradient and cost is not limited to the above inequality. Abert and Nikolov related two open problems to rank gradient as well: the fixed price problem and the multiplicativity of cost-1 problem [2].

We can easily prove that $RG(G, \{H_i\}) \geq RG_p(G) \geq \beta_1^{(2)}(G)$, which is a special case of the above inequality involving the rank gradient relative to a lattice of subgroups and p -gradient. Before proving the inequality we define explicitly p -gradient and rank gradient relative to a lattice of subgroups.

As stated earlier, we can define $RG_p(G) = RG(G_{\hat{p}})$. However, we can also give a more explicit definition of the p -gradient.

Definition. Let p be a prime. We define the p -gradient (also called mod- p homology gradient) of G by

$$RG_p(G) = \inf \frac{d_p(H) - 1}{[G : H]}$$

where $d_p(G) = d(G/[G, G]G^p)$ and the infimum is taken over all normal subgroups H such that $[G : H] = p^k$ for some $k \in \mathbb{Z}_{\geq 0}$.

The fact that $RG_p(G)$ as defined above is equal to $RG(G_{\widehat{p}})$ will be proved in Section 4. However, by using standard facts about profinite groups and pro- p groups, we can prove that the rank gradient and p -gradient of a pro- p group are equal.

Theorem 2.2. *If G is a (topologically) finitely generated pro- p group, then $RG_p(G) = RG(G)$.*

Proof. In a finitely generated pro- p group all finite index subgroups are open normal subgroups and have index a power of p [5]. Moreover, if H is a finite index subgroup of G , then H is also a finitely generated pro- p group. The Frattini subgroup of a finitely generated pro- p group H is $\Phi(H) = [H, H]H^p$ and by standard facts about finitely generated pro- p groups we have $d_p(H) = d(H/\Phi(H)) = d(H)$. Therefore,

$$RG_p(G) = \inf_{\substack{H \trianglelefteq G \\ [G:H]=p^k}} \frac{d_p(H) - 1}{[G : H]} = \inf_{[G:H] \leq \infty} \frac{d(H) - 1}{[G : H]} = RG(G).$$

□

Remark. We note that Nikolov and Segal proved Serre's conjecture on finitely generated profinite groups. That is, in a finitely generated profinite group all finite index subgroups are open [13].

Definition. 1. The rank gradient relative to a lattice $\{H_i\}$ of finite index subgroups is defined as

$$RG(G, \{H_i\}) = \inf_i \frac{d(H_i) - 1}{[G : H_i]}.$$

2. The p -gradient relative to a lattice $\{H_i\}$ of normal subgroups of p -power index is defined as

$$RG_p(G, \{H_i\}) = \inf_i \frac{d_p(H_i) - 1}{[G : H_i]}.$$

Often, the lattice is a descending chain of subgroups. In this case, we have the following useful lemma.

Lemma 2.3. *Let G be a finitely generated group. If $\{H_i\}_1^\infty$ is a descending chain of finite index subgroups and $\{K_i\}_1^\infty$ is a descending chain of normal subgroups of p -power index, then*

$$\left\{ \frac{d(H_i) - 1}{[G : H_i]} \right\}_1^\infty \quad \text{and} \quad \left\{ \frac{d_p(K_i) - 1}{[G : K_i]} \right\}_1^\infty$$

are non-increasing sequences. Further

$$RG(G, \{H_i\}) = \lim_{i \rightarrow \infty} \frac{d(H_i) - 1}{[G : H_i]} \quad \text{and} \quad RG_p(G, \{K_i\}) = \lim_{i \rightarrow \infty} \frac{d_p(K_i) - 1}{[G : K_i]}.$$

Proof. Since G is finitely generated and H_i is of finite index, then by the Schreier Index Formula we know that H_i is finitely generated and $d(H_{i+1}) - 1 \leq (d(H_i) - 1)[H_i : H_{i+1}]$ for each i . This implies that for each i ,

$$\frac{d(H_{i+1}) - 1}{[G : H_{i+1}]} \leq \frac{(d(H_i) - 1)[H_i : H_{i+1}]}{[G : H_{i+1}]} = \frac{d(H_i) - 1}{[G : H_i]}.$$

Therefore,

$$RG(G, \{H_i\}) = \inf_i \frac{d(H_i) - 1}{[G : H_i]} = \lim_{i \rightarrow \infty} \frac{d(H_i) - 1}{[G : H_i]}.$$

The corresponding result for p -gradient is proved similarly using the fact that for a finitely generated group G and a normal subgroup of p -power index K , the inequality $d_p(K) - 1 \leq (d_p(G) - 1)[G : K]$ holds. This can be proved using the Schreier Index Formula for finitely generated pro- p groups. \square

Proposition 2.4. *Let G be an infinite finitely presented residually- p group (that is, the intersection of all normal subgroups of p -power index is trivial). Let $\{H_i\}$ be an infinite strictly descending chain of normal subgroups of p -power index. Then $RG(G, \{H_i\}) \geq RG_p(G) \geq \beta_1^{(2)}(G)$.*

Proof. Since G is residually- p , for every infinite strictly descending chain of normal subgroups of p -power index, $\{H_i\}$, there exists another such chain $\{H'_i\}$ with trivial intersection such that H'_i is contained in H_i for all i . By the proof of Lemma 2.3 we have that $\frac{d(H_i) - 1}{[G : H_i]} \geq \frac{d(H'_i) - 1}{[G : H'_i]}$ for each i . Therefore, $RG(G, \{H_i\}) \geq RG(G, \{H'_i\})$.

Thus, it suffices to prove the result in the case when $\{H_i\}$ has trivial intersection. For every i we have $d(H_i) \geq d_p(H_i) \geq rk(H_i^{ab})$, which implies that $\frac{d(H_i)}{[G : H_i]} \geq \frac{d_p(H_i)}{[G : H_i]} \geq \frac{rk(H_i^{ab})}{[G : H_i]}$. By Theorem 2.1, taking the limit of the above inequality yields $RG(G, \{H_i\}) \geq RG_p(G, \{H_i\}) \geq \beta_1^{(2)}(G)$. By definition $RG_p(G, \{H_i\}) \geq RG_p(G) = \inf_{\{H_i\} \text{ chains}} RG_p(G, \{H_i\})$. Thus, we have $RG(G, \{H_i\}) \geq RG_p(G) \geq \beta_1^{(2)}(G)$. \square

Groups that have p -gradient > 0 are "big." In particular a group G is said to be p -large if it contains a normal subgroup of p -power index that maps onto a non-abelian free group. Lackenby proved the following result [9].

Theorem 2.5. (Lackenby) *Let G be a finitely presented group, and let p be a prime. Consider the following series of finite index normal subgroups: $H_0 = G$, $H_{i+1} = [H_i, H_i]H_i^p$. Suppose that $RG_p(G, \{H_i\}) > 0$. Then G is p -large.*

3 Rational Rank Gradient Values

We provide some useful theorems concerning the rank gradient of finite groups and how the rank gradient of a finite index subgroup relates to the rank gradient of a group.

Theorem 3.1. *Let G be a finite group. Then $RG(G) = -\frac{1}{|G|}$.*

Proof. If G is finite, then $\{1\}$ is a finite index subgroup of G and $\frac{d(\{1\})-1}{[G:\{1\}]} = \frac{-1}{|G|}$. By Lemma 2.3, $RG(G) = \frac{-1}{|G|}$. \square

Theorem 3.2. *Let G be a finitely generated group and let H be a finite index subgroup of G . Then $RG(G) = \frac{RG(H)}{[G:H]}$.*

Proof. Let $K \leq H \leq G$. Then $[G : K]$ is finite if and only if $[H : K]$ is finite and $[G : K] = [G : H][H : K]$. We have $\frac{d(K)-1}{[G:K]} = \frac{1}{[G:H]} \frac{d(K)-1}{[H:K]}$ and thus

$$\inf_{[G:K] < \infty} \frac{d(K)-1}{[G:K]} \leq \inf_{\substack{[G:K] < \infty \\ K \leq H}} \frac{d(K)-1}{[G:K]} = \frac{1}{[G:H]} \inf_{[H:K] < \infty} \frac{d(K)-1}{[H:K]}.$$

Therefore, $RG(G) \leq \frac{RG(H)}{[G:H]}$.

Now, $\{K \leq H \mid [H : K] < \infty\} = \{H \cap K \mid [G : K] < \infty\}$ Note that $[K : H \cap K]$ is finite and so $[G : H \cap K] = [G : K][K : H \cap K]$ and by

Schreier Index Formula $d(H \cap K) - 1 \leq (d(K) - 1)[K : H \cap K]$. Therefore, $\frac{d(H \cap K) - 1}{[G : H \cap K]} \leq \frac{(d(K) - 1)[K : H \cap K]}{[G : H \cap K]} = \frac{d(K) - 1}{[G : K]}$. This gives us

$$\begin{aligned} RG(G) &= \inf_{[G:K] < \infty} \frac{d(K) - 1}{[G : K]} \geq \inf_{[G:H \cap K] < \infty} \frac{d(H \cap K) - 1}{[G : H \cap K]} \\ &= \inf_{[H:H \cap K] < \infty} \frac{d(H \cap K) - 1}{[G : H \cap K]} = \frac{1}{[G : H]} \inf_{[H:H \cap K] < \infty} \frac{d(H \cap K) - 1}{[H : H \cap K]} \\ &= \frac{RG(H)}{[G : H]}. \end{aligned}$$

Therefore, $RG(G) \geq \frac{RG(H)}{[G:H]}$. \square

Lemma 3.3. *Let F be a non-abelian free group of finite rank and let p be a prime number. Then $RG(F) = RG_p(F) = \text{rank}(F) - 1$.*

Proof. First, for any free group F and any prime p , we know $d_p(F) = d(F)$. Let H be a finite index subgroup of F . Since H is free, $d_p(H) = d(H)$ and by Schreier Index Formula, $d(H) - 1 = (d(F) - 1)[F : H]$, which implies in this case $d_p(H) - 1 = (d_p(F) - 1)[F : H]$. Therefore,

$$RG(F) = \inf_{[F:H] < \infty} \frac{d(H) - 1}{[F : H]} = \inf_{[F:H] < \infty} (d(F) - 1) = \text{rank}(F) - 1,$$

$$RG_p(F) = \inf_{\substack{H \trianglelefteq F \\ p\text{-power}}} \frac{d_p(H) - 1}{[F : H]} = \inf_{\substack{H \trianglelefteq F \\ p\text{-power}}} (d_p(F) - 1) = \text{rank}(F) - 1.$$

\square

As the following proposition shows, it is not difficult to produce groups with rational rank gradient. The question remains open of whether an irrational number can be the rank gradient of some finitely generated group. We will show later that for every prime p , every positive real number is the p -gradient for some finitely generated group.

Proposition 3.4. *Let $q > 0 \in \mathbb{Q}$. There exists a finitely presented group G such that $RG(G) = q$.*

Proof. Write $q = \frac{m}{n}$. Let F_{m+1} be a non-abelian free group of rank $m + 1$ and let A be any group of order n . Consider $G = F_{m+1} \times A$. Let $\phi : G \rightarrow A$ be the projection onto the second component and let $H = \ker \phi$. Then $F_{m+1} \simeq H$ and $[G : H] = n$. By Theorem 3.2, $RG(G) = \frac{RG(H)}{[G:H]} = \frac{m}{n}$. \square

4 p -Gradient and Pro- p Completions

In this section, we will prove that a group and its pro- p completion have the same p -gradient. To get this result, we must gather some facts about profinite groups.

Let G be a finitely generated group. We will denote the pro- p completion of G for some prime p by $G_{\hat{p}}$. We let $d(G)$ denote the minimal number of abstract generators of a group G if the group is not profinite and the minimal number of topological generators if the group is profinite. If a group is profinite we will use “finitely generated” to mean topologically finitely generated. The reader is referred to any standard text in profinite groups for the basic results used in this section [5], [18].

When dealing with pro- p completions of a group, it is often convenient to assume that the group is residually- p since then the group will imbed in its pro- p completion. To show why this type of assumption will not influence any of our results about the p -gradient, we give the following lemma.

Definition. Let G be a group and p a prime. Let \mathcal{N} , the p -residual of G , be the intersection of all normal subgroups of p -power index in G . The p -residualization of G is the quotient G/\mathcal{N} . Note that the p -residualization of G is isomorphic to the image of G in its pro- p completion $G_{\hat{p}}$ and is residually- p .

Lemma 4.1. *Let G be a group and fix a prime p . Let \tilde{G} be the p -residualization of G . Then*

1. $RG_p(G) = RG_p(\tilde{G})$.
2. $G_{\hat{p}} \simeq \tilde{G}_{\hat{p}}$.

Proof. 1. We first note that every normal subgroup of p -power index in G contains \mathcal{N} . Therefore, there is a bijective correspondence between normal subgroups of p -power index in \tilde{G} and normal subgroups of p -power index in G . Let the correspondence be $\tilde{H} \mapsto H$ with $\tilde{H} \leq \tilde{G}$ and $H \leq G$. Then $[\tilde{G} : \tilde{H}] = [G : H]$ and $\tilde{H} \simeq H/\mathcal{N}$. Therefore, $\tilde{H}/[\tilde{H}, \tilde{H}](\tilde{H})^p \simeq H/([H, H]H^p\mathcal{N}) \simeq H/[H, H]H^p$ since $[H, H]H^p$ is a p -power index normal subgroup of G and thus contains \mathcal{N} . Therefore, $d_p(\tilde{H}) = d_p(H)$.

Since there is a bijection $\tilde{H} \mapsto H$ between all normal subgroups of p -power index in \tilde{G} and G , and for each we have $[\tilde{G} : \tilde{H}] = [G : H]$ and $d_p(\tilde{H}) = d_p(H)$, then $RG_p(\tilde{G}) = RG_p(G)$.

2. By the proof of (1) above, $\tilde{G}/\tilde{H} \simeq (G/\mathcal{N})/(H/\mathcal{N}) \simeq G/H$ and the result follows by the inverse limit definition of pro- p completion and the universal property of pro- p completions. \square

The following fact is well known.

Proposition 4.2. *Let G be a group and p a prime number. The set of subnormal subgroups of p -power index form a base of neighborhoods of the identity for the pro- p topology on G .*

With the following proposition, we will be able to prove that a group and its pro- p completion have the same p -gradient. Parts of this proposition can be found in an exercise in [5].

Proposition 4.3. *Let G be a finitely generated group and p a prime. Let $\varphi : G \rightarrow G_{\hat{p}}$ be the natural map from G to its pro- p completion. Let H be a normal subgroup of p -power index of G . The following hold.*

1. $\varphi(H) = \varphi(G) \cap \overline{\varphi(H)}$.
2. $\overline{\varphi} : G/H \rightarrow G_{\hat{p}}/\overline{\varphi(H)}$ given by $\overline{\varphi}(xH) = \varphi(x)\overline{\varphi(H)}$ is an isomorphism.
3. There exists an index preserving bijection between normal subgroups of p -power index in G and open normal subgroups of $G_{\hat{p}}$.
4. $\overline{\varphi(H)} \simeq H_{\hat{p}}$ as pro- p groups.
5. $RG(G_{\hat{p}}) = \frac{RG(H_{\hat{p}})}{[G : H]}$.

Proof. Parts (1)-(3) are proved in Proposition 3.2.2 of Ribes and Zalesskii [16]. We prove parts (4) and (5).

4) For notational simplicity, assume G is residually- p and thus φ is injective. The case of G not residually- p is proved similarly. We only need to show that the pro- p topology on G induces the pro- p topology on the subspace H of G . By Proposition 4.2, subnormal subgroups of p -power index in G form a base for the pro- p topology. If K is subnormal of p -power index in H it implies that K is subnormal of p -power index in G . This implies that the subspace topology on H and the pro- p topology are the same. Therefore, $\overline{\varphi(H)} \simeq H_{\hat{p}}$ as pro- p groups.

5) By (2) and (4) we know $G/H \simeq G_{\hat{p}}/H_{\hat{p}}$ and therefore $[G : H] = [G_{\hat{p}} : H_{\hat{p}}]$. Thus, by Theorem 3.2 we have $RG(G_{\hat{p}}) = \frac{RG(H_{\hat{p}})}{[G_{\hat{p}} : H_{\hat{p}}]} = \frac{RG(H_{\hat{p}})}{[G : H]}$. \square

We are now ready to prove the relationship between the p -gradient of a group and its pro- p completion.

Theorem 4.4. *Let G be a finitely generated group and p a fixed prime. Let $G_{\widehat{p}}$ be the pro- p completion of G . Then $RG_p(G) = RG_p(G_{\widehat{p}})$.*

Proof. We start by assuming the case that G is residually- p . Then there is an injective map $\varphi : G \rightarrow G_{\widehat{p}}$ such that $\varphi(G) = G$ is dense in $G_{\widehat{p}}$. Therefore, if G is finitely generated then it implies that $G_{\widehat{p}}$ is finitely generated as a pro- p group. In a finitely generated pro- p group all finite index subgroups are open normal subgroups and have index a power of p [5]. By Proposition 4.3.3 we know that $H \rightarrow \overline{H}$ is an index preserving bijection between the normal subgroups of p -power index in G and the normal subgroups of p -power index in $G_{\widehat{p}}$.

Since $RG_p(G) = RG_p(G_{\widehat{p}})$ if $d_p(H) = d_p(\overline{H})$ for all p -power index normal subgroups $H \trianglelefteq G$, it suffices to show

$$H/[H, H]H^p \simeq \overline{H}/[\overline{H}, \overline{H}]\overline{H}^p.$$

Now, by Proposition 4.3.4 we know $\overline{H} \simeq H_{\widehat{p}}$ as pro- p groups. Also, H is residually- p and thus the natural map $\psi : H \rightarrow H_{\widehat{p}}$ is injective. Therefore, by Proposition 4.3.2 we have $H/[H, H]H^p \simeq H_{\widehat{p}}/\text{closure}_{H_{\widehat{p}}}([H, H]H^p)$. Since $[H, H]H^p \subseteq H$ it implies $\overline{[H, H]H^p} \subseteq \overline{H}$. Therefore,

$$H_{\widehat{p}}/\text{closure}_{H_{\widehat{p}}}([H, H]H^p) \simeq \overline{H}/\overline{H} \cap \overline{[H, H]H^p} \simeq \overline{H}/\overline{[H, H]H^p}.$$

Thus, $H/[H, H]H^p \simeq \overline{H}/\overline{[H, H]H^p}$ and so it remains to show

$$[\overline{H}, \overline{H}]\overline{H}^p = \overline{[H, H]H^p}.$$

“ \supseteq ” Clearly, $\Phi(\overline{H}) = [\overline{H}, \overline{H}]\overline{H}^p \supseteq [H, H]H^p$ with $\Phi(\overline{H})$ the Frattini subgroup of \overline{H} . We note that $\Phi(\overline{H})$ is open and thus closed. Thus, $[\overline{H}, \overline{H}]\overline{H}^p \supseteq \overline{[H, H]H^p}$.

“ \subseteq ” For ease of notation let $B = [H, H]H^p$. We know $\overline{H}/\overline{B} \simeq H/B$. Thus,

$$\frac{[\overline{H}, \overline{H}]\overline{H}^p}{\overline{B}} \simeq \left[\frac{\overline{H}}{\overline{B}}, \frac{\overline{H}}{\overline{B}} \right] \left(\frac{\overline{H}}{\overline{B}} \right)^p \simeq \left[\frac{H}{B}, \frac{H}{B} \right] \left(\frac{H}{B} \right)^p \simeq \frac{[H, H]H^p}{B} = 1.$$

Therefore, we have $[\overline{H}, \overline{H}]\overline{H}^p \subseteq \overline{[H, H]H^p}$.

For a residually- p group, we have $RG_p(G) = RG_p(G_{\widehat{p}})$. However, if G is not residually- p , let \tilde{G} be the p -residualization of G . Then by Lemma 4.1 we know $RG_p(G) = RG_p(\tilde{G})$ and $\tilde{G}_{\widehat{p}} \simeq G_{\widehat{p}}$. Therefore, $RG_p(G) = RG_p(\tilde{G}) = RG_p(\tilde{G}_{\widehat{p}}) = RG_p(G_{\widehat{p}})$. \square

The above theorem provides some very useful corollaries.

Corollary 4.5. *Let G be a finitely generated group and p a fixed prime. Let $G_{\widehat{p}}$ be the pro- p completion of G . Then $RG_p(G) = RG_p(G_{\widehat{p}}) = RG(G_{\widehat{p}})$.*

Corollary 4.6. *If G is a finite group, then $RG_p(G) = -\frac{1}{|G_{\widehat{p}}|}$.*

Proof. If G is finite, then so is $G_{\widehat{p}}$ and thus $RG_p(G) = RG_p(G_{\widehat{p}}) = RG(G_{\widehat{p}}) = -\frac{1}{|G_{\widehat{p}}|}$ by Theorem 3.1. \square

Theorem 4.7. *Fix a prime p and let G be a finitely generated group. Assume $H \leq G$ is a p -power index subnormal subgroup. Then $RG_p(G) = \frac{RG_p(H)}{[G:H]}$.*

Proof. Since H is subnormal of p -power index, then there exists subgroups $H = H_0, H_1, H_2, \dots, H_n = G$ such that $H_i \trianglelefteq H_{i+1}$ and $[H_{i+1} : H_i]$ is a p -power. We will induct on the subnormal length of H . Assume H is 1-subnormal and thus H is normal in G .

By Proposition 4.3.5 we know $RG(G_{\widehat{p}}) = \frac{RG(H_{\widehat{p}})}{[G:H]}$ and by Corollary 4.5 we have $RG_p(G) = RG(G_{\widehat{p}}) = \frac{RG(H_{\widehat{p}})}{[G:H]} = \frac{RG_p(H)}{[G:H]}$. Now, assume H is n -subnormal. Then H_{n-1} is normal in G and therefore, $RG_p(G) = \frac{RG_p(H_{n-1})}{[G:H_{n-1}]}$. Also, H is $(n-1)$ -subnormal in H_{n-1} and thus by induction $RG_p(H_{n-1}) = \frac{RG_p(H)}{[H_{n-1}:H]}$. Therefore, $RG_p(G) = \frac{1}{[G:H_{n-1}]} \frac{RG_p(H)}{[H_{n-1}:H]} = \frac{RG_p(H)}{[G:H]}$. \square

5 Groups With Arbitrary p -Gradient Values

In this section we will prove the main result, that is, we construct a finitely generated group Γ with $RG_p(\Gamma) = \alpha$ for each $\alpha > 0 \in \mathbb{R}$. To prove this, we need some technical results.

The following lemma is similar to Lemma 2.3 of Osin [14] concerning deficiency of a finitely presented group.

Lemma 5.1. *Let G be a finitely generated group and fix a prime p . Let x be some non-trivial element of G . Let H be a finite index normal subgroup of G such that $x^m \in H$, but no smaller power of x is in H . Let $\pi : G \rightarrow G/\langle x^m \rangle^G$ be the standard projection homomorphism.*

1. *If T is a right transversal for $\langle x \rangle H$ in G , then $\langle x^m \rangle^G = \langle tx^mt^{-1} \mid t \in T \rangle^H$.*
2. *If $H = \langle Y \mid R \rangle$, then $\pi(H) = \langle Y \mid R \cup \{tx^mt^{-1} \mid t \in T\} \rangle$.*

$$3. |T| = \frac{[G : H]}{m}.$$

$$4. \text{ If } \mathfrak{q}(H) = \frac{d_p(H)}{[G : H]}, \text{ then } \mathfrak{q}(\pi(H)) \geq \mathfrak{q}(H) - \frac{1}{m}.$$

Proof. Since x^m is in H , then $[\pi(G) : \pi(H)] = [G : H]$.

1. This is a standard computation.
2. This holds by (1) and the fact that $\pi(H) = H/(H \cap \langle x^m \rangle^G) = H/\langle x^m \rangle^G$, since $x^m \in H$ and H is normal in G .
3. Since $H \subseteq \langle x \rangle H \subseteq G$, then $[G : H] = [G : \langle x \rangle H][\langle x \rangle H : H]$. Therefore, $|T| = [G : \langle x \rangle H] = \frac{[G : H]}{[\langle x \rangle H : H]}$. Since $x^m \in H$ but no smaller power of x is in H , then we know that $V = \{1, x, x^2, \dots, x^{m-1}\}$ is a transversal for H in $\langle x \rangle H$ and thus $[\langle x \rangle H : H] = m$. Therefore, $|T| = \frac{[G : H]}{m}$.
4. We first note that (2) and (3) imply that a presentation for $\pi(H)$ is obtained from a presentation for H by adding in $\frac{[G : H]}{m}$ relations. Now,

$$\mathfrak{q}(\pi(H)) \geq \mathfrak{q}(H) - \frac{1}{m}$$

if and only if

$$d_p(\pi(H)) \geq d_p(H) - \frac{[G : H]}{m}.$$

If H has presentation $H = \langle Y \mid R \rangle$ then $\pi(H)$ has presentation $\pi(H) = \langle Y \mid R \cup \{tx^mt^{-1} \text{ for all } t \in T\} \rangle$. For notational simplicity let $C = \{[y_1, y_2] \mid y_1, y_2 \in Y\}$. Then,

$$H/([H, H]H^p) = \langle Y \mid R, C, w^p \text{ for all } w \in F(Y) \rangle$$

and

$$\pi(H)/([\pi(H), \pi(H)]\pi(H)^p) = \langle Y \mid R, C, w^p \text{ for all } w \in F(Y), \\ tx^mt^{-1} \text{ for all } t \in T \rangle.$$

Therefore, a presentation for $\pi(H)/([\pi(H), \pi(H)]\pi(H)^p)$ is obtained from a presentation for $H/([H, H]H^p)$ by adding in $\frac{[G : H]}{m}$ relations.

Note: For any group G , we can consider $G/([G, G]G^p)$ as a vector space over \mathbb{F}_p and therefore $d_p(G)$ is the dimension of this vector space.

Therefore, by our above remarks, we see that $\pi(H)/([\pi(H), \pi(H)]\pi(H)^p)$ is just a vector space satisfying $\frac{[G:H]}{m}$ more equations than the vector space $H/([H, H]H^p)$. Thus $d_p(\pi(H)) \geq d_p(H) - \frac{[G:H]}{m}$. \square

Using the above lemma we have the following lower bound for the p -gradient when taking the quotient by the normal subgroup generated by an element raised to a p -power.

Theorem 5.2. *Let G be a finitely generated group, p some fixed prime, and $x \in G$. Then $RG_p(G/\langle\langle x^{p^k} \rangle\rangle) \geq RG_p(G) - \frac{1}{p^k}$.*

Proof. Case 1: There exists a normal subgroup H_0 of p -power index such that the order of x in G/H_0 is at least p^k .

Assume the order of x in G/H_0 is $p^t \geq p^k$. Let \overline{H} be a normal subgroup of p -power index in $\overline{G} = G/\langle\langle x^{p^k} \rangle\rangle$. Let $H \leq G$ be the full preimage of \overline{H} . Then H is a p -power index normal subgroup in G which contains $\langle\langle x^{p^k} \rangle\rangle$ and in particular $x^{p^t} = (x^{p^k})^{p^{t-k}} \in H$. Let $L_H = H \cap H_0$. Then L_H is a normal subgroup in G such that $x^{p^t} \in L_H$, $L_H \subseteq H$, and the order of x in G/L_H is p^t . Note that L_H is normal and of p -power index in G since both H and H_0 are normal and of p -power index. Thus by Lemma 5.1, we have

$$\mathfrak{q}(\overline{H}) \geq \mathfrak{q}(\overline{L_H}) \geq \mathfrak{q}(L_H) - \frac{1}{p^t} \geq \mathfrak{q}(L_H) - \frac{1}{p^k},$$

which by definition is $\geq RG_p(G) - \frac{1}{p^k}$. Therefore $\mathfrak{q}(\overline{H}) \geq RG_p(G) - \frac{1}{p^k}$. Thus $RG_p(G/\langle\langle x^{p^k} \rangle\rangle) \geq RG_p(G) - \frac{1}{p^k}$.

Case 2: For every normal subgroup H of p -power index, the order of x in G/H is less than p^k .

We will actually show that $RG_p(G/\langle\langle x^{p^k} \rangle\rangle) = RG_p(G)$ in this case. There exists an $\ell < k$ such that $x^{p^\ell} \in H$ for every normal subgroup H of p -power index in G . Then x^{p^ℓ} is in the kernel of natural map from G to its pro- p completion $\varphi : G \rightarrow G_{\widehat{p}}$. Therefore, $x^{p^k} = (x^{p^\ell})^{p^{k-\ell}} \in \ker \varphi$. Let $M = \langle\langle x^{p^k} \rangle\rangle$. Then $M \subseteq \ker \varphi$. This implies that there is a bijective correspondence between all normal subgroup of p -power index in G and G/M given by $N \rightarrow N/M$. Thus since $G/N \simeq (G/M)/(N/M)$ for all such N , then using the inverse limit definition of pro- p completion we have $G_{\widehat{p}} \simeq (G/M)_{\widehat{p}}$ as pro- p groups. Therefore, $RG_p(G/\langle\langle x^{p^k} \rangle\rangle) = RG_p(G/M) = RG_p((G/M)_{\widehat{p}}) = RG_p(G_{\widehat{p}}) = RG_p(G)$. \square

Remark. The above theorem was independently stated and proved using slightly different language and a different method by Barnea and Schlage-Puchta (Theorem 4 in [4]).

Corollary 5.3. *Let G be a finitely generated group, p some fixed prime, and let $x \in G$. Then $RG_p(G/\langle\langle x \rangle\rangle) \geq RG_p(G) - 1$.*

5.1 p -Gradient and Direct Limits

Let (I, \leq) be a totally ordered set with smallest element 0 and let $\{G_i \mid \pi_{ij}\}$ be a direct system of finitely generated groups with surjective homomorphisms $\pi_{ij} : G_i \rightarrow G_j$ for every $j \geq i \in I$.

Let $G_\infty = \varinjlim G_i$ be the direct limit of this direct system. Let $\pi_i : G_i \rightarrow G_\infty$ be the map obtained from the direct limit. Since all of the maps in the direct system are surjective, then so are the π_i . Let $G = G_0$.

We can define another direct system $\{M_i \mid \mu_{ij}\}$ over the same indexing set I , where $M_i = G$ for each i and μ_{ij} is the identity map. The direct limit of this set is clearly $G = \varinjlim M_i$ and the map obtained from the direct limit $\mu_i : M_i \rightarrow G_\infty$ is the identity map.

A homomorphism $\Phi : \{M_i \mid \mu_{ij}\} \rightarrow \{G_i \mid \pi_{ij}\}$ is by definition a family of group homomorphisms $\varphi_i : M_i \rightarrow G_i$ such that $\varphi_j \circ \mu_{ij} = \pi_{ij} \circ \varphi_i$ whenever $i \leq j$. Then Φ defines a unique homomorphism $\varphi = \varinjlim \varphi_i : \varinjlim M_i \rightarrow \varinjlim G_i$ such that $\varphi \circ \mu_i = \pi_i \circ \varphi_i$ for all $i \in I$ [3].

We see that $\varphi_i : G \rightarrow G_i$ is the map π_{0i} in this case. It is clear that $\varphi = \varinjlim \varphi_i$. Since each φ_i is surjective, it implies that $\ker \varphi_i \subseteq \ker \varphi_j$ for every $j \geq i$. In this situation we have

$$\ker \varphi = \varinjlim \ker \varphi_i = \bigcup_{i \in I} \ker \varphi_i.$$

Let $H \leq G$ be a subgroup. For every i , let $H_i = \varphi_i(H)$.

Lemma 5.4. *We keep the notation defined above. Fix a prime p . For each $K \trianglelefteq G_\infty$ of p -power index, there exists an $H' \trianglelefteq G$ of p -power index such that*

1. $K = \varinjlim H'_i$.
2. $[G_\infty : K] = \lim_{i \in I} [G_i : H'_i]$.
3. $d_p(K) = \lim_{i \in I} d_p(H'_i)$.

Proof. Let $K \trianglelefteq G_\infty$ be a p -power index normal subgroup. Since $\varphi : G \rightarrow G_\infty$ is surjective then $G_\infty \simeq G/\ker \varphi$. Let $H' = \varphi^{-1}(K)$. Then H' is normal in G and since $K \simeq H'/\ker \varphi$ we have $[G_\infty : K] = [G : H']$ and so H' is of p -power index.

1. $K = \varphi(H') = \varinjlim \varphi_i(H') = \varinjlim H'_i$.
2. Since each $\varphi_i : G \rightarrow G_i$ is surjective, we have $G_i \simeq G/\ker \varphi_i$ and since H' contains $\ker \varphi$, then H' contains $\ker \varphi_i$ for each i . Thus, $H'_i \simeq H'/\ker \varphi_i$. Therefore for every i ,

$$G_i/H'_i \simeq G/H' \simeq G_\infty/K.$$

Thus, for every i we have $[G_\infty : K] = [G_i : H'_i]$.

3. For any group A , let $Q(A) = A/[A, A]A^p$. We know $K \simeq H'/\ker \varphi$ and $H'_i \simeq H'/\ker \varphi_i$ and therefore,

$$Q(K) \simeq H'/[H', H'](H')^p \ker \varphi \simeq Q(H')/M$$

where $M = [H', H'](H')^p \ker \varphi/[H', H'](H')^p$, and

$$Q(H'_i) \simeq H'/[H', H'](H')^p \ker \varphi_i \simeq Q(H')/M_i$$

where $M_i = [H', H'](H')^p \ker \varphi_i/[H', H'](H')^p$. Since $\ker \varphi_i \subseteq \ker \varphi_j$ for each $j \geq i$ then $M_i \subseteq M_j$ for each $j \geq i$. Now, $Q(H')$ is finitely generated abelian and torsion and therefore is finite. Thus $Q(H')$ can only have finitely many non-isomorphic subgroups. Since $\{M_i\}$ is an ascending set of subgroups, there must exist an $n \in I$ such that $M_i = M_n$ for every $i \geq n$. Since $\ker \varphi_i \subseteq \ker \varphi_j$ for each $j \geq i$ and $\bigcup \ker \varphi_i = \ker \varphi$, we know that $M_i \subseteq M_j$ for every $j \geq i$ and $\bigcup M_i = M$. Therefore, $M = \bigcup M_i = M_n$. Thus for each $i \geq n$ we have $M = M_i$.

Therefore, $Q(K) \simeq Q(H'_i)$ for each $i \geq n$ and therefore $d_p(K) = d_p(H'_i)$ for each $i \geq n$. Thus $d_p(K) = \lim_{i \in I} d_p(H'_i)$. \square

The following lemma is similar to Pichot's related result for L^2 -Betti numbers where he considers convergence is in the space of marked groups [15].

Lemma 5.5. *For each prime p , $\limsup RG_p(G_i) \leq RG_p(G_\infty)$.*

Proof. Fix a prime p . Let $K \trianglelefteq G_\infty$ be a normal subgroup of p -power index. By Lemma 5.4 we obtain the subgroups H' and H'_i for each i . Now,

$$\limsup RG_p(G_i) = \limsup \inf_{\substack{N \trianglelefteq G_i \\ p\text{-power}}} \frac{d_p(N) - 1}{[G_i : N]} \leq \limsup \frac{d_p(H'_i) - 1}{[G_i : H'_i]}$$

and by Lemma 5.4

$$\limsup \frac{d_p(H'_i) - 1}{[G_i : H'_i]} = \lim_{i \in I} \frac{d_p(H'_i) - 1}{[G_i : H'_i]} = \frac{d_p(K) - 1}{[G_\infty : K]}.$$

Therefore, for each $K \trianglelefteq G_\infty$ of p -power index, we have $\limsup RG_p(G_i) \leq \frac{d_p(K) - 1}{[G_\infty : K]}$. This implies $\limsup RG_p(G_i) \leq RG_p(G_\infty)$. \square

5.2 The Main Result

We are ready to prove the main result that every nonnegative real number is realized as the p -gradient of some finitely generated group.

Theorem 5.6. (Main Result) *For every real number $\alpha > 0$ and any prime p , there exists a finitely generated group Γ such that $RG_p(\Gamma) = \alpha$.*

Proof. Fix a prime p and a real number $\alpha > 0$. Let F be the free group on $[\alpha] + 1$ generators. Let

$$\Lambda = \{G \mid F \text{ surjects onto } G, G \text{ is residually-}p, \text{ and } RG_p(G) \geq \alpha\}.$$

We know Λ is not empty since F is in Λ . We can partially order Λ by $G_1 \succcurlyeq G_2$ if there is an epimorphism from G_1 to G_2 , denoted $G_1 \twoheadrightarrow G_2$. We know this order is antisymmetric since each group in this set is Hopfian.

Let $\mathcal{C} = \{G_i\}$ be a chain in Λ . Then each chain forms a direct system of groups over a totally ordered indexing set. We can extend any chain so that it starts with the element $F = G_0$. Let $G_\infty = \varinjlim G_i$.

By Lemma 5.5 we know $RG_p(G_\infty) \geq \limsup RG_p(G_i) \geq \alpha$. If we let \tilde{G}_∞ be the p -residualization of G_∞ , then by Lemma 4.1, $RG_p(\tilde{G}_\infty) = RG_p(G_\infty)$. Therefore, $RG_p(\tilde{G}_\infty) \geq \alpha$, and \tilde{G}_∞ is residually- p . Moreover, for each i we have $G_i \twoheadrightarrow G_\infty$ and in particular $F \twoheadrightarrow G_\infty \twoheadrightarrow \tilde{G}_\infty$. Thus $\tilde{G}_\infty \in \Lambda$ and $G_i \succcurlyeq \tilde{G}_\infty$ for each i . Thus each chain \mathcal{C} in Λ has a lower bound in Λ and therefore by Zorn's Lemma Λ has a minimal element, call it Γ .

Since Γ and its p -residualization $\tilde{\Gamma}$ have the same p -gradient and Γ surjects onto $\tilde{\Gamma}$, it implies that $\tilde{\Gamma} \in \Lambda$ and $\Gamma \succcurlyeq \tilde{\Gamma}$. Thus Γ must be residually- p , otherwise $\tilde{\Gamma}$ contradicts the minimality of Γ .

Note: Γ does not have finite exponent.

If Γ had finite exponent then since Γ is finitely generated and residually finite it must be finite by the positive solution to the Restricted Burnside Problem [19]. This would imply $RG_p(\Gamma) < 0$ by Corollary 4.6. This contradicts that Γ is in Λ .

Therefore, Γ is a finitely generated residually- p group with infinite exponent such that $RG_p(\Gamma) \geq \alpha$.

Claim: $RG_p(\Gamma) = \alpha$.

Assume not. Then there exists a $k \in \mathbb{N}$ such that $RG_p(\Gamma) - \frac{1}{p^k} \geq \alpha$. Since Γ is residually- p , the order of every element is a power of p and since Γ has infinite exponent, there exists $x \in \Gamma$ whose order is greater than p^k .

Consider $\Gamma' = \Gamma / \langle\langle x^{p^k} \rangle\rangle$. Since $x^{p^k} \neq 1$ it implies that $\Gamma' \not\cong \Gamma$. By Theorem 5.2 we have that $RG_p(\Gamma') \geq RG_p(\Gamma) - \frac{1}{p^k} \geq \alpha$. If Γ' is not residually- p , replace it with its p -residualization, which we know will have the same p -gradient. Then $\Gamma' \in \Lambda$ and $\Gamma \not\approx \Gamma'$, which contradicts the minimality of Γ . \square

We can strengthen the result of Theorem 5.6 without much effort.

Theorem 5.7. *Fix a prime p . For every real number $\alpha > 0$ there exists a finitely generated residually- p torsion group Γ such that $RG_p(\Gamma) = \alpha$.*

Proof. Barnea and Puchta showed in Corollary 5 of [4], that for any $\alpha > 0$ there exists a torsion group \mathcal{G} with $RG_p(\mathcal{G}) \geq \alpha$. If we now apply the construction given in Theorem 5.6 replacing the free group F with the p -residualization of \mathcal{G} , the resulting group Γ will be torsion, residually- p and $RG_p(\Gamma) = \alpha$. \square

We note that Y. Barnea and J.C. Schlage-Puchta [4] proved a result similar to Theorem 5.7 (inequality instead of equality) albeit in a slightly different way.

6 Applications

The construction given in Theorem 5.6 has a few immediate applications. First, we note that Theorem 5.7 gives a known counter example to the General Burnside Problem. The second application is more general and shows that there exist uncountably many pairwise non-commensurable groups that are finitely generated, infinite, torsion, non-amenable, and residually- p .

The following theorem was proved by Abert, Jaikin-Zapirain, and Nikolov in [1]. Lackenby first proved the result for finitely presented groups in [8].

Theorem 6.1. (*Abert, Jaikin-Zapirain, Nikolov*) *Finitely generated infinite amenable groups have rank gradient zero with respect to any normal chain with trivial intersection.*

As a simple corollary, we provide a corresponding, albeit weaker, result concerning p -gradient.

Corollary 6.2. *If $RG_p(G) > 0$ for any prime p , then G is not amenable.*

Proof. Let G be a finitely generated group with $RG_p(G) > 0$. Let \tilde{G} be the p -residualization of G . Then $0 < RG_p(G) = RG_p(\tilde{G})$. Now, let $\{H_i\}$ be a descending chain of normal subgroups of p -power index in \tilde{G} which intersect in the identity. Then

$$0 < RG_p(\tilde{G}) \leq \inf_i \frac{d_p(H_i) - 1}{[\tilde{G} : H_i]} \leq \inf_i \frac{d(H_i) - 1}{[\tilde{G} : H_i]} = RG(\tilde{G}, \{H_i\}).$$

Therefore, by Theorem 6.1 we know \tilde{G} is not amenable. This implies that G is not amenable since a quotient of an amenable group is amenable. \square

The application of the construction used in Theorem 5.6 concerning commensurable groups is given below.

Definition. Two groups are called commensurable if they have isomorphic subgroups of finite index.

The following lemma is straitforward.

Lemma 6.3. *Fix a prime p . Let G be a p -torsion group (every element has order a power of p). Then every finite index subgroup $H \leq G$ is subnormal of p -power index.*

Theorem 6.4. *There exist uncountably many pairwise non-commensurable groups that are finitely generated, infinite, torsion, non-amenable, and residually- p .*

Proof. Let p be a fixed prime number. By Theorem 5.7 we know that for every real number $\alpha > 0$ there exists a finitely generated residually- p infinite torsion group, Γ , such that $RG_p(\Gamma) = \alpha$. By Corollary 6.2 we know that these groups are all non-amenable. Since each of these groups is residually- p and torsion, we know that they are actually all p -torsion. Thus every subgroup of finite index in these groups is subnormal of p -power index.

By Theorem 4.7 if any two of these groups are commensurable, then the p -gradient of each group is a rational multiple of the other. Since there are uncountably many positive real numbers, which are not rational multiples of each other, we have our result. \square

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